On the Problem of a Finitistic Quantum Field Theory

F. Winterberg

Desert Research Institute, University of Nevada System, Reno, Nevada 89506

A finitistic quantum field theory, as a model for a unified theory of elementary particles is proposed, making the assumption that all spatial and temporal distances can only assume integer values of a fundamental length and time, for which we have chosen the Planck length and Planck time. To satisfy the condition of causality in its quantized version, the theory must be exactly nonrelativistic, because only then can the concept that points in space are separated by multiples of a fundamental length be formulated in an invariant way. The theory is formulated as a partial finite difference equation, invariant under translations and rotations in space, and translations in time, and can be expressed as a partial differential equation of infinite order. The theory is free of all divergencies, has a positive definite metric in Hilbert space and therefore no ghost states. Possessing a fundamental length, chosen to be equal the Planck length, the theory has an inbuilt cut-off at the Planck energy. For energies sufficiently below the Planck energy, the theory can be approximated by the previously described Planck aether model, which can be viewed as a superfluid consisting of Planck masses, leading to special relativity as a dynamic symmetry in the asymptotic limit of low energies.

1. Introduction

To describe an element of reality, a physical theory should be expressed by a countable set of values. Already a classical field theory described by a partial differential equation contains metaphysical elements, because it requires for the computation of the field at some future time the knowledge of the field strength at an infinite number of points in space. In principle that includes values of the field strength at all points in space, whether they are expressed by rational or irrational numbers. This metaphysical element plays a minor role in classical physics, where it limits the accuracy in the prediction of the future through the inaccuracy of the initial conditions, but it leads to fundamental difficulties in quantum field theories. In classical field theories only those solutions of the field equation have to be considered which behave like analytic functions, whereas in quantized field theories the much larger number of nonanalytic functions, like the Dirichlet function which for all rational numbers has the value 1 and vanishes for all irrational numbers, have to be considered as well. Such functions are under normal conditions of no importance in classical physics, where they do not represent real physical processes, but as intermediate virtual states they cannot be excluded from quantum theory. It was first pointed out by Heisenberg [1] that the divergence problems of quantized field theories are likely to have here their root.

Historically, the first attempts made to solve the problem of the divergencies was through the introduction of a cut-off length limiting the otherwise divergent integrals. Such a procedure is quite successful in solid state physics theories, which describe the solid as a field and where the lattice constant serves as a natural cut-off length. The reason why it does not work in relativistic quantum field theories is the indefinite metric of the Minkowski space-time, which makes it impossible to formulate the property of proximity between two points in space in a relativistically invariant way. A relativistic field theory containing a cut-off parameter violates causality, because through a Lorentz-transformation the time sequence of cause and effect can be reversed. In an attempt to overcome this problem, Heisenberg introduced the idea of an indefinite metric in Hilbert space, by dividing the Hilbert space into a Hilbert space I containing all states possessing a mass smaller than a limiting upper mass M_g, and a Hilbert space II, containing all the other states, like those described by the above-mentioned nonanalytic functions. According to Heisenberg, the rules of quantum mechanics should then only apply to Hilbert space I. However, in order to avoid violation of causality, the limiting mass postulated by Heisenberg would have to be made arbitrarily large, because an upper finite mass it equivalent to a smallest length. Therefore, one would have to make the transition $M_g \to \infty$, and at the same time keep the

indefinite metric in Hilbert space. It appears that this leads to unphysical ghost states like in other regularization methods.

A different noteworthy attempt to solve the divergence problem was first made by Bopp, who introduced the hypothesis of a lattice space [2]. Since the Hilbert space has there a finite number of dimensions, the introduction of a Hilbert space II becomes superfluous. However, in order to avoid the above-mentioned problem of causality violation in a relativistic theory with a cut-off, Bopp believes in departures from special relativity in the small, very much as they occur in the large through the presence of gravitational fields. And because quantum electrodynamics suggests a cut-off at the gravitational radius of the electron, he believes that gravity may provide a natural cut-off. Bopp does not make a specific proposal how this idea would have to be implemented, but it appears doubtful that it could possibly work, because the microscopic space-time structure of general relativity is Minkowskian, and for this reason does not help in formulating the proximity property in an invariant way.

To solve the divergence problem of relativistic quantum field theories, I had suggested that in the limit of the Planck length the correct fundamental kinematic symmetry of nature is the Galilei group, and that the fundamental field equation envisioned by Heisenberg must be exactly nonrelativistic. Special relativity would there emerge as a dynamic symmetry for energies below the Planck scale, in a way as it was proposed prior to Einstein by Lorentz and Poincaré. In support of this conjecture one may cite the growing experimental evidence suggesting that physics becomes unified at an energy of $\sim 10^{15}$–10^{16} GeV, which on a logarithmic scale is surprisingly close to the Planck energy of $\sim 10^{19}$ GeV, reached at the Planck length of $\sim 10^{-33}$ cm.*

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* Quantum gravity suggests a foam-like space-time structure at the Planck length, leading to the absurd conclusion that widely separated points in space-time can bridge these distances by interacting through "wormholes". But even without quantization, general relativity predicts that the time evolution of its solutions end in a singularity, something which is clearly unphysical and which shows that in the course of gravitational collapse Einstein's field equations must break down, most likely in approaching the Schwarzschild radius. Since the Planck length is the Schwarzschild radius of the zero point vacuum energy, the occurrence of such singularities already on the classical level raises doubts about the claim of quantum gravity for a foam-like space-time structure at the Planck length.

In an attempt to formulate a Heisenberg-type non-relativistic fundamental field theory, a nonlinear Schrödinger wave equation was used as a model [3]. It contains the Planck mass as a parameter, admitting both positive and negative values, with two fields $\psi_\pm$ representing this two-valuedness. In its quantized version it leads to the operator field equation

$$i\hbar \frac{\partial \psi_\pm}{\partial t} = \mp \frac{\hbar^2}{2m_{\mathrm{p}}} \nabla^2 \psi_\pm$$
$$+ 2\hbar c\, r_{\mathrm{p}}^2 (\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-)\psi_\pm , \quad (1.1)$$

$$[\psi_\pm(r)\,\psi_\pm^\dagger(r')] = \delta(r - r'),$$
$$[\psi_\pm(r)\,\psi_\pm(r')] = [\psi_\pm^\dagger(r)\,\psi_\pm^\dagger(r')] = 0, \quad (1.2)$$

where $r_{\mathrm{p}} = \sqrt{\hbar G/c^3}$ and $m_{\mathrm{p}} = \sqrt{\hbar c/G}$ are the Planck length and Planck mass, with $m_{\mathrm{p}}\, r_{\mathrm{p}}\, c = \hbar$ and $G m_{\mathrm{p}}^2 = \hbar c$. Making a Hartree approximation, whereby the field operators are replaced by their expectation values, the resulting two-component nonlinear Schrödinger equation resembles the Landau-Ginzburg equation of superfluidity. This model describes the vacuum in its groundstate by a two-component superfluid composed of densely packed positive and negative Planck masses. In this approximation the discretization of the model into a superfluid composed of Planck masses permits the introduction of a cut-off which, as in a Debye model, can be chosen to by equal to the Planck length. Because this superfluid has a distinguished reference system in which it is at rest, I have called it the Planck aether. In this model the Planck aether assumes the role of the fundamental field in the spirit of Heisenberg's theory, and from which all waves, and particles are made of. It was shown, that from this model not only Maxwell's and Einstein's field equations can be derived, but even Dirac spinors. And for energies well below the Planck energy special relativity emerges as a derived dynamic symmetry. The phenomenon of charge receives in the model the simple explanation to result from the quantum mechanical zero point fluctuations of Planck masses bound in vortex filaments of the superfluid Planck aether.

With the Galilei group as the fundamental kinematic symmetry, separating space from time, the proximity of points can now be defined in an invariant way, permitting the introduction of a fundamental length. We note that the field equation (1.1) does not make use of this possibility, since its eigenvalues at high energies (omitting the nonlinear coupling term)

are given by

$$E_\pm = \pm \hbar^2 k^2 / 2 m_p \qquad (1.3)$$

Because the spectrum (1.3) is not limited, (1.1) still includes all those wave fields expressed by nonanalytic functions, which according to Heisenberg should be excluded. But whereas Heisenberg could not introduce in his relativistic field equation a fundamental length limiting the dimensionality in Hilbert space, this certainly is possible in a nonrelativistic field equation. In case of the field equation (1.1) one can always introduce a cut-off as it is done in a Debye model, but from a purely formalistic point of view such an approach is unsatisfactory. In a fully consistent theory the cut-off should be a consequence of the field equation and should not be introduced as an ad hoc assumption.

2. Finitistic Field Equation

An object of reality requires a measurement, which in an abstract way can be seen as a counting procedure involving natural numbers. All distances in space and time should therefore only be measured in integer multiples of a fundamental length and time.* In accordance with this principle, a field theory describing physical reality should have the form of a finite difference equation. If the fundamental length is chosen to be equal the Planck length r_p, the fundamental time interval would have to be $t_p = r_p / c$. For a field equation of the type (1.1), the differentials have to be replaced by finite differences.** Following a procedure outlined by Madelung [5], we can express the finite difference quotient of a function $y = f(x)$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+l/2) - f(x-l/2)}{l} \qquad (2.1)$$

through the finite difference operator

$$f(x+h) = e^{h\,d/dx} f(x)$$

$$= f(x) + h \frac{df(x)}{dx} + \frac{h^2}{2!} \frac{d^2 f(x)}{dx^2} + \dots \qquad (2.2)$$

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* We would like to remark that in the framework of quantum gravity, v. Borzeszkowski and Treder have come to a similar conclusion [4].
** It must be emphasized that the introduction of a fundamental interval as here proposed is not the same as a lattice space, because a lattice space has a discrete group of translations and rotations.

With $h = l/2$, we therefore find

$$\frac{\Delta y}{\Delta x} = \frac{\sinh\left[(l/2)\,d/dx\right]}{l/2} f(x). \qquad (2.3)$$

In a similar way we can define a Riemann integral average y-value $\bar{y}$ by

$$\bar{y} = (f(x+l/2) + f(x-l/2))/2$$

$$= \cosh\left[(l/2)\,d/dx\right] f(x). \qquad (2.4)$$

In the limit $l \to 0$, (2.3) becomes dy/dx and (2.4) y. Putting $d/dx = \partial$, we may introduce the operators

$$\Delta_0 = \cosh\left[(l/2)\,\partial\right],$$

$$\Delta_1 = (2/l) \sinh\left[(l/2)\,\partial\right] \qquad (2.5)$$

such that

$$\frac{\Delta y}{\Delta x} = \Delta_1 f(x), \quad \bar{y} = \Delta_0 f(x) \qquad (2.6)$$

and furthermore

$$\Delta_1 = (2/l)^2 \, d\Delta_0/d\partial. \qquad (2.7)$$

Both operators are also solutions of

$$[d^2/d\partial^2 - (l/2)^2] \, \Delta(\partial) = 0. \qquad (2.8)$$

In the context of the fundamental field equation (1.1), we therefore may make the substitution (putting $l = t_p$)

$$\frac{\partial \psi}{\partial t} \to \Delta_1 \psi = \frac{2}{t_p} \sinh\left[(t_p/2) \frac{\partial}{\partial t}\right] \psi, \qquad (2.9)$$

whereby the energy operator becomes

$$E = i\hbar \Delta_1. \qquad (2.10)$$

For the replacement of the differential operator of the space part, we must find the three-dimensional generalization D_0 and D_1 of the operators Δ_0 and Δ_1.

As shown in Appendix I, these operators are

$$D_0 = \frac{\sinh\left[\sqrt{3}\,(r_p/2)\,\partial\right]}{\sqrt{3}\,(r_p/2)\,\partial} \qquad (2.11)$$

and

$$D_1 = \left(\frac{2}{r_p}\right)^2 \left\{ \cosh\left[\sqrt{3}\,(r_p/2)\,\partial\right] - \frac{\sinh\left[\sqrt{3}\,(r_p/2)\,\partial\right]}{\sqrt{3}\,(r_p/2)\,\partial} \right\} \frac{\partial_i}{\partial^2}. \qquad (2.12)$$

We therefore have to make the replacement

$$\nabla^2 \to D_1^2 \qquad (2.13)$$

with the momentum operator $p = (\hbar/i)\, \partial/\partial q$ to be replaced by

$$p = \frac{\hbar}{i}\, D_1. \qquad (2.14)$$

To insure the integrity of the classical Poisson bracket relation $\{q, p\} = 1$, a change in the momentum operator p must be accompanied by a change in the position operator q. If in the commutation relation

$$p\,q - q\,p = \hbar/i, \qquad (2.15)$$

p is expressed by (2.14), the position operator is then given by *

$$q = \left(\frac{\partial_i}{D_1}\right) r \qquad (2.16)$$

which in the limit $r_p \to o$ results in $q \to r$.

The commutation relation (1.2) is therefore changed into

$$[\psi_\pm (r)\, \psi_\pm^\dagger (r')] = D(|r - r'|), \qquad (2.17)$$

where in accordance with the finite difference calculus, $D(|r - r'|)$ is a generalized three-dimensional delta function for which

$$\lim_{(r_p \to 0)} D(|r - r'|) = \delta(|r - r'|). \qquad (2.18)$$

In Leibniz's operator notation we may formally put

$$\int = \frac{1}{d}, \qquad (2.19)$$

resulting in the operator equation

$$1/(d/dx) = (1/d)\, dx = \int dx. \qquad (2.20)$$

Applied to a one-dimensional delta-function one has

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = \frac{1}{d}\, \delta(x)\, dx = \frac{1}{d/dx}\, \delta(x) = 1. \quad (2.21)$$

Since for finite difference operations d/dx is replaced by D_1, the equation corresponding to (2.21) is

$$\frac{1}{D_1}\, D(x) = 1 \qquad (2.22)$$

and for the three-dimensional D-function occurring in (2.17)

$$\frac{1}{D_1^3}\, D(|r - r'|) = 1. \qquad (2.23)$$

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* For the connection of the position operator with the position eigenfunction, see Appendix II.

If we replace in D_1 the operator symbol $\partial_i = \partial/\partial x_i$ by the equivalent operator symbol $(\int \partial x_i)^{-1}$, and thereafter expand D_1^{-3} into a power series of the operator $\int \partial x_i$, (2.23) then consists of an infinite number of integrations (very much as D_1 consists of an infinite number of differentiations).

In (1.1) the nonlinear selfinteraction terms result from a delta-function type contact interaction. If this interaction is replaced by the finite difference D-function, and if the infinite number of integrations are carried out in accordance with (2.23), the selfinteraction terms remain unchanged.

Replacing in (1.1) $\partial/\partial t$ by Δ_1 and ∇ by D_1, we thus arrive at the following finitistic field equation

$$i\hbar \Delta_1 \psi_\pm = \mp \frac{\hbar^2}{2m_p}\, D_1^2 \psi_\pm$$
$$+ 2\hbar c\, r_p^2\, [\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-]\, \psi_\pm. \qquad (2.24)$$

Putting $\partial/\partial t = \partial_t$ and hence

$$\Delta_1 = \left(\frac{2}{t_p}\right)^2 \frac{d\Delta_0}{d\partial_t} \qquad (2.25)$$

and with (I.9) we can write instead of (2.24)

$$i\left(\frac{2}{t_p}\right)\frac{d\Delta_0}{d\partial_t}\, \psi_\pm = \mp \left(\frac{2}{r_p}\right)^2 \left(\frac{dD_0}{d\partial}\, \frac{\partial_i}{\partial}\right)^2 \psi_\pm$$
$$+ r_p^3\, [\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-]\, \psi_\pm. \qquad (2.26)$$

In this form, only the Planck length and the Planck time, but not the Planck mass, enters the finitistic field equation. To speak of Planck masses becomes then meaningful only if one approximates (2.26) respectively (2.24) by (1.1). In the limit $r_p \to 0$, $t_p \to 0$, (2.24) goes over into (1.1) and which therefore is an approximation of (2.24) for $r/r_p \gg 1$, $t/t_p \gg 1$.

In the quantum mechanical equation of motion for an operator F

$$\frac{dF}{dt} = \frac{i}{\hbar}(HF - FH) \equiv \frac{i}{\hbar}[H, F], \qquad (2.27)$$

where H is the Hamilton operator, the r.h.s. remains unchanged, because the Poisson bracket for any dynamical quantity can be reduced to a sum of Poisson brackets for position and momentum, whereas in the l.h.s. the operator d/dt has to be replaced by Δ_1. The equation of motion (2.27) is therefore changed into

$$\Delta_1 F = \frac{i}{\hbar}[H, F] \qquad (2.28)$$

It can be shown that the proposed finitistic field theory is also fully consistent with the Lagrange-formalism, provided we replace everywhere the space and time differentiation operators by finite difference operators. For an unquantized classical field, the operators ψ and $\psi^\dagger$ are replaced by the functions ϕ and ϕ^*, and one has for the Lagrange-density

$$\mathscr{L}_\pm = i\hbar\,\phi_\pm^* \,\Delta_1\,\phi_\pm \mp \frac{\hbar^2}{2m_p}(D_1\phi_\pm^*)\cdot(D_1\phi_\pm)$$
$$\mp 2\hbar c\,r_p^2\,[\tfrac{1}{2}\phi_\pm^*\phi_\pm - \phi_\pm^*\phi_\mp]\,\phi_\pm^*\phi_\pm. \quad (2.29)$$

Variation with regard to ϕ^*, according to,

$$\frac{\partial\mathscr{L}_\pm}{\partial\phi_\pm^*} - D_1\frac{\partial\mathscr{L}_\pm}{\partial(D_1\phi_\pm^*)} = 0 \quad (2.30)$$

results in the classical field equation

$$i\hbar\,\Delta_1\phi_\pm = \mp\frac{\hbar^2}{2m_p}D_1^2\phi_\pm$$
$$+ 2\hbar c\,r_p^2\,[\phi_\mp^*\phi_+ - \phi_\pm^*\phi_-]\,\phi_\pm. \quad (2.31)$$

If the variation is carried out with regard to ϕ, according to

$$\frac{\partial\mathscr{L}_\pm}{\partial\phi_\pm} - D_1\frac{\partial\mathscr{L}_\pm}{\partial(D_1\phi_\pm)} - \Delta_1\frac{\partial\mathscr{L}_\pm}{\partial(\Delta_1\phi_\pm)} = 0 \quad (2.32)$$

one obtains the conjugate complex equation of (2.31), making the replacement $\phi\to\phi^*$, $\Delta_1\phi\to-\Delta_1\phi^*$.

The momentum density canonically conjugate to ϕ is

$$\pi_\pm = \frac{\partial\mathscr{L}_\pm}{\partial(\Delta_1\phi_\pm)} = i\hbar\,\phi_\pm^* \quad (2.33)$$

and hence the Hamilton density

$$\mathscr{H}_\pm = \pi_\pm\,\Delta_1\phi_\pm - \mathscr{L}_\pm$$
$$= \mp\frac{i\hbar}{2m_p}(D_1\pi_\pm)\cdot(D_1\phi_\pm)$$
$$\mp 2ic\,r_p^2\,[\tfrac{1}{2}\phi_\pm^*\phi_\pm - \phi_\mp^*\phi_\mp]\,\pi_\pm\phi_\pm$$
$$= \pm\frac{\hbar^2}{2m_p}(D_1\phi_\pm^*)\cdot(D_1\phi_\pm)$$
$$\pm 2\hbar c\,r_p^2\,[\tfrac{1}{2}\phi_\pm^*\phi_\pm - \phi_\mp^*\phi_\mp]\,\phi_\pm^*\phi_\pm. \quad (2.34)$$

We are now in a position to show that the unquantized and quantized field equations formally agree. According to (2.28) the operator ψ must obey the equation of motion

$$i\hbar\,\Delta_1\psi = [\psi, H], \quad (2.35)$$

where

$$H = D_1^{-3}\mathscr{H} \quad (2.36)$$

with D_1^{-3} the finite difference volume integration operator satisfying (2.23). We therefore must have

$$i\hbar\,\Delta_1\psi_\pm = \left[\psi_\pm, \pm D_1^{-3\prime}\frac{\hbar^2}{2m_p}(D_1'\psi_\pm^{\dagger\prime})\cdot(D_1'\psi_\pm')\right]$$
$$+ [\psi_\pm, \pm D_1^{-3\prime}2\hbar c\,r_p^2(\tfrac{1}{2}\psi_\pm^{\dagger\prime}\psi_\pm' - \psi_\mp^{\dagger\prime}\psi_\mp')\psi_\pm^{\dagger\prime}\psi_\pm']. \quad (2.37)$$

To evaluate these commutators we have to use (2.17). In the first commutator we obtain by partial (finite difference operator) integration

$$[\psi, D_1^{-3\prime}(D_1'\psi^{\dagger\prime})\cdot(D_1'\psi')] = -[\psi, D_1^{-3\prime}\psi^{\dagger\prime}D_1'^2\psi']$$
$$= -D_1^{-3\prime}[\psi',\psi^{\dagger\prime}]D_1'^2\psi'$$
$$= -D_1^{-3\prime}D_1'^2\psi'D(|\mathbf{r}-\mathbf{r}'|)$$
$$= -D_1^2\psi. \quad (2.38)$$

In evaluating the second commutator of (2.37) we can take the operator $\psi_-^\dagger\psi_-$ as a numerical (c-number) function with regard to the operators $\psi_+^\dagger$, ψ_+ and likewise the operator $\psi_+^\dagger\psi_+$ as a numerical function with regard to the operators $\psi_-^\dagger$, ψ_-. For these terms we have expressions in the integrand of the form

$$\psi\psi^{\dagger\prime}\psi' - \psi^{\dagger\prime}\psi'\psi = \psi'(\psi\psi^{\dagger\prime} - \psi^{\dagger\prime}\psi)$$
$$= \psi'D(|\mathbf{r}-\mathbf{r}'|). \quad (2.39)$$

Multiplying (that is finite-difference integrating) (2.39) by $D_1^{-3\prime}$ leads to ψ. For the remaining terms the integrands have the form

$$\psi\psi^{\dagger\prime}\psi'\psi^{\dagger\prime}\psi' - \psi^{\dagger\prime}\psi'\psi^{\dagger\prime}\psi'\psi$$
$$= \psi^{\dagger\prime}\psi\psi'\psi^{\dagger\prime}\psi' + D(|\mathbf{r}-\mathbf{r}'|)\psi'\psi^{\dagger\prime}\psi' - \psi^{\dagger\prime}\psi'\psi^{\dagger\prime}\psi'\psi$$
$$= \psi^{\dagger\prime}\psi'\psi\psi^{\dagger\prime}\psi' - \psi^{\dagger\prime}\psi'\psi^{\dagger\prime}\psi'\psi + D(|\mathbf{r}-\mathbf{r}'|)\psi'\psi^{\dagger\prime}\psi'$$
$$= \psi^{\dagger\prime}\psi'[\psi\psi^{\dagger\prime}\psi' - \psi^{\dagger\prime}\psi'\psi] + D(|\mathbf{r}-\mathbf{r}'|)\psi'\psi^{\dagger\prime}\psi'$$
$$= \psi^{\dagger\prime}\psi'\psi'D(|\mathbf{r}-\mathbf{r}'|) + D(|\mathbf{r}-\mathbf{r}'|)\psi'\psi^{\dagger\prime}\psi'$$
$$= 2\psi^{\dagger\prime}\psi'\psi'D(|\mathbf{r}-\mathbf{r}'|). \quad (2.40)$$

Multiplying (2.40) by $D_1^{-3\prime}$ then leads to $2\psi^\dagger\psi\psi$. Inserting the results of (2.38)–(2.40) into (2.37), one obtains the operator field equation (2.24), and which shows that the classical and quantum equations have the same form.

Finally, we may prove that our finitistic field theory conserves the number of particles, and which can be

seen as the conservation of the number of "Planck-ions". With the particle number operator

$$N_\pm = D_1^{-3} \psi_\pm^\dagger \psi_\pm \qquad (2.41)$$

one has to show that

$$i\hbar \Delta_1 N_\pm = [N_\pm, H_\pm] = 0. \qquad (2.42)$$

As before, $\psi_-^\dagger \psi_-$ can be treated as a c-number with regard to the operators $\psi_+^\dagger, \psi_+$ and vice versa. In these terms $\psi_-^\dagger \psi_-$ (respectively $\psi_+^\dagger \psi_+$) therefore acts like an external potential. Since it is well known that for a nonrelativistic field theory without interaction, but in the presence of an external potential, the particle number is conserved, we have only to show that this is also true if the nonlinear selfinteraction term is included. In the commutator it leads to integrals with integrands of the form (and which can be transformed using (2.17)):

$$\psi^\dagger \psi \psi^{\dagger\prime} \psi^\prime \psi^{\dagger\prime} \psi^\prime - \psi^{\dagger\prime} \psi^\prime \psi^{\dagger\prime} \psi^\prime \psi^\dagger \psi$$
$$= \psi^\dagger \psi^{\dagger\prime} \psi \psi^\prime \psi^{\dagger\prime} \psi^\prime + \psi^\dagger D(|\boldsymbol{r}-\boldsymbol{r}^\prime|) \psi^\prime \psi^{\dagger\prime} \psi^\prime$$
$$\qquad - \psi^{\dagger\prime} \psi^\prime \psi^{\dagger\prime} \psi^\prime \psi^\dagger \psi$$
$$= \psi^{\dagger\prime} \psi^\dagger \psi^\prime \psi \psi^{\dagger\prime} \psi^\prime + \psi^\dagger D(|\boldsymbol{r}-\boldsymbol{r}^\prime|) \psi^\prime \psi^{\dagger\prime} \psi^\prime$$
$$\qquad - \psi^{\dagger\prime} \psi^\prime \psi^{\dagger\prime} \psi^\prime \psi^\dagger \psi$$
$$= \psi^{\dagger\prime} \psi^\prime \psi^\dagger \psi \psi^{\dagger\prime} \psi^\prime - \psi^{\dagger\prime} D(|\boldsymbol{r}-\boldsymbol{r}^\prime|) \psi \psi^{\dagger\prime} \psi^\prime$$
$$\qquad + \psi^\dagger D(|\boldsymbol{r}-\boldsymbol{r}^\prime|) \psi^\prime \psi^{\dagger\prime} \psi^\prime$$
$$\qquad - \psi^{\dagger\prime} \psi^\prime \psi^{\dagger\prime} \psi^\prime \psi^\dagger \psi$$
$$= \psi^{\dagger\prime} \psi^\prime [\psi^\dagger \psi \psi^{\dagger\prime} \psi^\prime - \psi^{\dagger\prime} \psi^\prime \psi^\dagger \psi]$$
$$\qquad + \psi^\dagger D(|\boldsymbol{r}-\boldsymbol{r}^\prime|) \psi^\prime \psi^{\dagger\prime} \psi^\prime$$
$$\qquad - \psi^{\dagger\prime} D(|\boldsymbol{r}-\boldsymbol{r}^\prime|) \psi \psi^{\dagger\prime} \psi^\prime. \qquad (2.43)$$

In (2.43) the last two terms cancel upon integration (multiplication with D_1^{-3}). The first term is zero as well, because it can be reduced to a term which would arise in (2.42) in the presence of an externally applied potential. There the integrand contains the term $\psi^\dagger \psi \psi^{\dagger\prime} \psi^\prime - \psi^{\dagger\prime} \psi^\prime \psi^\dagger \psi$, and which is of the same form as the bracket in (2.43). It therefore follows that the number of Planckions is conserved. For this reason it is impossible for two Planckions (of opposite mass) to annihilate each other, which would be possible in the presence of a frictional force. In this latter case the potential would have to be complex, and two Planck-ions of opposite sign could come to mutual rest. With-out it, two Planckions of opposite sign encountering a head-on collision simply pass through each other.

3. Maximum Energy and Momentum

As we had already pointed out, for energies $E \ll m_p c^2$ we can approximate the finite difference field equation (2.24) by (1.1) and the commutation relation (2.17) by (1.2). In this approximation the field equation leads to the phonon-roton spectrum below the Planck scale. In the vicinity of $E \simeq m_p c^2$, where the approximation becomes invalid, departures from the energy spectrum of (1.1) are to be expected, and with it a limitation in the dimensionality of the Hilbert space. To study this departure, we may omit the non-linear selfcoupling term. If the free field is limited in the dimensionality of the Hilbert space, the selfcou-pling can not alter this fact.

The finite difference field equation

$$i\hbar \Delta_1 \psi = -\frac{\hbar^2}{2m_p} D_1^2 \psi \qquad (3.1)$$

has for a wave-field $\psi = \psi(x, t)$ the form

$$\frac{2i\hbar}{t_p} \sinh\left[\frac{t_p}{2} \frac{\partial}{\partial t}\right] \psi = -\frac{8 m_p c^2}{r_p^2}$$
$$\cdot \left\{ \cosh\left[\sqrt{3}(r_p/2)\partial/\partial x\right] - \frac{\sinh\left[\sqrt{3}(r_p/2)\partial/\partial x\right]}{\sqrt{3}(r_p/2)\partial/\partial x} \right\}^2$$
$$\cdot \frac{1}{(\partial/\partial x)^2} \psi. \qquad (3.2)$$

For a plane wave

$$\psi = A e^{i(kx - \omega t)}. \qquad (3.3)$$

(3.2) leads to the dispersion relation

$$\sin(\omega t_p/2) = \frac{3}{(\sqrt{3}(r_p/2)k)^2}$$
$$\cdot \left\{ \frac{\sin[\sqrt{3}(r_p/2)k]}{\sqrt{3}(r_p/2)k} - \cos[\sqrt{3}(r_p/2)k] \right\}^2. \qquad (3.4)$$

Putting

$$x = \sqrt{3}(r_p/2)k, \qquad (3.5)$$

one can write instead of (3.4)

$$\qquad (3.6)$$
$$\sqrt{3} \sin(\omega t_p/2) = f(x), \quad f(x) = \frac{3}{x}\left[\frac{\sin x}{x} - \cos x\right].$$

In the limit $x \to 0$, one has $f(x) \to x$, and for $x \to \infty$, $f(x) \to 0$. The function $f(x)$ has a maximum at $x \simeq 2.1$, where $f(2.1) \simeq 1.3$. We therefore find that

$$k_{max} \simeq 2.4/r_p \qquad (3.7)$$

and that therefore

$$\sin(\omega_{max} t_p/2) \simeq 0.57. \qquad (3.8)$$

The maximum energy is then computed with the energy operator (2.10):

$$E_{max} = \frac{2\hbar}{t_p} \sin(\omega_{max} t_p/2)$$

$$= 2 m_p c^2 \sin(\omega_{max} t_p/2) \simeq 1.14 m_p c^2. \qquad (3.9)$$

From (3.8) we also have

$$\omega_{max} t_p \simeq 1.22 \qquad (3.10)$$

and hence

$$\frac{\omega_{max}}{k_{max}} \simeq 0.56 c. \qquad (3.11)$$

4. Concluding Remarks

The assumption that the fundamental symmetry in nature is the Galilei group, broken below the Planck scale into the Lorentz group, enabled us to formulate a finitistic field theory. In this theory, elements of reality are measured values of the field at multiples of the Planck length and Planck time. Reality therefore becomes countable (as in a computer). Classical field theory requires of the computation of the evolution in time, the value of the field at an infinite number of points, thereby giving the field metaphysical properties outside of the countable world physics is supposed to be. The finitistic field theory presented here does not have this problem. It only requires the measurement of a very large but countable number of values. Whereas in classical field theory the field must be measured at all possible points in space and time, but where the differential equation to compute the evolution of the field is of finite order, the finitistic field theory presented here requires that the field has to be measured at a countable number of points number of space-time, with a countable number of initial conditions expressed as finite difference quotients.

The proposed finitistic field theory also seems to cast some light on the particle-wave dualism of physics. In nonrelativistic quantum field theories the particle number is always conserved, which in relativistic quantum field theories is only true for fields without interaction. With energy and momentum conserved and representative for the wave nature of matter, we may say that in a relativistic quantum field theory the "wave-property" is conserved. Nonrelativistic quantum theory starts from the particle picture, whereas relativistic quantum field theory takes its start from classical wave fields. The justification of the particle concept in relativistic field theories is obtained through the irreducible representation of the Poincaré group, which however is valid only for free particles in which case the Hamilton operator commutes with the particle number operator. In the finitistic field theory presented here, there are neither particles nor waves, but rather finite displacements, expressed through operators coupled in a fundamental equation. Only for energies below the Planck scale can the theory be approximated by interacting Planck masses, and further below by a world of waves, appearing to be continuous but involving a very large number of Planck masses. One can now see why the Galilei group is somehow representative for the particle property, and the Lorentz group for the wave property of matter. Assuming that all physical objects are held together by wave fields, with the wave velocity equal the velocity of light, Lorentz invariance immediately follows as an approximate dynamic symmetry. This symmetry is exact, if the medium in which the wave propagates is an absolute continuum. It would break down for a medium made up of discrete objects, if the wave length would become that small. If the discreteness of the medium is given by the Planck length, this would imply a departure from Lorentz invariance at the Planck energy $\sim 10^{19}$ GeV. It is therefore no wonder that Lorentz invariance is so extremely well satisfied at the much smaller energies accessible to man.

Appendix I

To obtain the difference operator in an N-dimensional space, we have generalize the differential equation for the one-dimensional "average value" operator $\Delta_0(\partial)$

$$[\mathrm{d}^2/\mathrm{d}\partial^2 - (l/2)^2] \Delta_0(\partial) = 0 \qquad (\mathrm{I}.1)$$

with the condition

$$\lim_{l \to 0} \Delta_0(\partial) = 1 \qquad (\mathrm{I}.2)$$

to an N-dimensional space.

Calling the generalized N-dimensional "average value" operator Δ_0^N, it would have to satisfy the partial

differential equation

$$\left[\sum_{i=1}^{N} \frac{\partial^2}{\partial (\partial_i)^2} - N \left(\frac{l}{2} \right)^2 \right] \Delta_0^N = 0, \qquad (I.3)$$

where for $N = 3$, $\partial_i = \left\{ \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\} = \{ \partial_x, \partial_y, \partial_z \}$.

Because Δ_0^N must be a scalar function, and because the only vector invariant which is a scalar is

$$\partial = \sqrt{ \sum_{i=1}^{N} \partial_i^2 }, \qquad (I.4)$$

one must have $\Delta_0^N = \Delta_0^N (\partial)$. One can therefore introduce into (I.3) N-dimensional polar coordinates, and obtains the ordinary differential equation

$$\left[\frac{1}{\partial^{N-1}} \frac{d}{d\partial} \left(\partial^{N-1} \frac{d}{d\partial} \right) - N \left(\frac{l}{2} \right)^2 \right] \Delta_0^N (\partial) = 0. \qquad (I.5)$$

The general solution of (I.5) can be expressed in terms of cylinder functions [6]. Having obtained the scalar operator function $\Delta_0^N (\partial)$, the generalized difference operator, invariant under arbitrary translations and rotations, is given by

$$\Delta_1^N = (2/l)^2 \, d\Delta_0^N / d\partial_i, \qquad (I.6)$$

leading for $N = 1$ to (2.7).

For the three-dimensional case, needed in our application, we call $\Delta_0^3 \equiv D_0$, and $\Delta_1^3 \equiv D_1$, and furthermore put $l \equiv r_p$. A solution of (I.5) for $N = 3$, with the property

$$\lim_{r_p \to 0} D_0 = 1 \qquad (I.7)$$

is then given by

$$D_0 = \frac{\sinh [\sqrt{3}\,(r_p/2)\,\partial]}{\sqrt{3}\,(r_p/2)\,\partial}. \qquad (I.8)$$

With

$$D_1 = \left(\frac{2}{r_p} \right)^2 \frac{dD_0}{d\partial_i} = \left(\frac{2}{r_p} \right)^2 \frac{dD_0}{d\partial} \frac{d\partial}{d\partial_i} = \left(\frac{2}{r_p} \right)^2 \frac{dD_0}{d\partial} \frac{\partial_i}{\partial} \qquad (I.9)$$

we obtain

$$D_1 = \left(\frac{2}{r_p} \right)^2 \left\{ \cosh [\sqrt{3}\,(r_p/2)\,\partial] - \frac{\sinh [\sqrt{3}\,(r_p/2)\,\partial]}{\sqrt{3}\,(r_p/2)\,\partial} \right\}$$
$$\cdot \frac{\partial_i}{\partial^2}. \qquad (I.10)$$

Expanding the bracket in (I.10) up to 3rd order terms, using

$$\cosh x = 1 + x^2/2 + \dots,$$
$$\sinh x = x + x^3/6 + \dots, \qquad (I.11)$$

we find that

$$\lim_{r_p \to 0} D_1 = \partial_i \qquad (I.12)$$

Appendix II

With $\psi_n (q)$ the (one dimensional) position eigenfunction and q the position operator, the eigenvalues q_n and eigenfunctions are determined by the equation

$$q \psi_n (q) = q_n \psi_n (q). \qquad (II.1)$$

If the position can be precisely measured one would have

$$\psi_n (q) = \delta (q - q_n). \qquad (II.2)$$

Inserting (II.2) into the l.h.s. of (II.1) and integrating from $q = -\infty$ to $q = +\infty$ one has

$$\int_{-\infty}^{+\infty} q \, \delta (q - q_n) \, dq = q_n. \qquad (II.3)$$

Using Leibniz's operator notation $\int = 1/d$ one can also write for (II.3)

$$\frac{dq}{d} q \, \delta (q - q_n) = q_n. \qquad (II.4)$$

For $q = q_n$ one has

$$\frac{dq}{d} \delta (q - q_n) = 1. \qquad (II.5)$$

Now, if the position eigenfunction is instead given by the generalized delta-function $D(q - q_n)$, and which was defined by (2.22) such that

$$\frac{1}{D_1} D(q - q_n) = 1, \qquad (II.6)$$

it follows by comparison with (II.4) and (II.5) that with the position eigenfunction given by $D(q - q_n)$, the position operator must be replaced by putting

$$q \to \frac{1}{D_1} \left(\frac{d}{dq} \right) q \qquad (II.7)$$

which is equivalent with (2.16).

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